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Remark on the Equations $\delta R^2/\delta g_{ij} = 0$

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It is shown that the 4-dimensional equations $\delta R^2/\delta g_{ij} = 0$ may be rewritten as 5-dimensional equations which are linear in the components of the Riemann tensor.

If R is the scalar curvature of an n-dimensional Riemann space V_n, the functional derivative of its square is given by

$$-\tfrac{1}{2}\delta R^2/\delta g_{ij} = R^{;ij} + RR^{ij} - g^{ij}(\Box R + \tfrac{1}{4}R^2)$$

The equations $\delta R^2/\delta g_{ij} = 0$ have occasionally been considered when $n = 4$ in the context of gravitational theory, and they may be written

$$R_{;ij} + RR_{ij} - \tfrac{1}{4}g_{ij}R^2 = 0 \tag{1a}$$

$$\Box R = 0 \tag{1b}$$

the second of these now being a consequence of the first. I call a set of equations "R-linear" or "R-nonlinear" according to whether it is or is not, respectively, linear in the components of the Riemann tensor. Thus (1) is R-nonlinear.

Now let $\bar{V}_5$ be a 5-dimensional Riemann space which is static with respect to x^5. If the coordinates are suitably chosen the metric of the $\bar{V}_5$ will take the generic form

$$ds^2 = \bar{g}_{\mu\nu}\,dx^\mu\,dx^\nu = \bar{g}_{ij}\,dx^i\,dx^j + \bar{g}_{55}(dx^5)^2 \tag{2}$$

where $\bar{g}_{\mu\nu,5} = 0$. (Greek and roman indices have the ranges 1–5 and 1–4, respectively.) All quantities defined on the $\bar{V}_5$ are distinguished by a bar, e.g., its Ricci tensor is $\bar{R}_{\mu\nu}$. Then the object of this note is to show that (when

$R \neq 0$) equations (1) can be written in the form of a set of 5-dimensional *R-linear* equations, viz.,

$$\bar{R}_{ij} - \tfrac{1}{4}\bar{g}_{ij}\bar{R} = 0 \tag{3}$$

It is understood throughout that the metric of the $\bar{V}_5$ has the generic form (2). To adapt it more closely to the purpose at hand, we write

$$V^{-2} := \bar{g}_{55} \qquad g_{ij} := V^{-2}\bar{g}_{ij}$$

and then

$$ds^2 = V^2 g_{ij}\, dx^i\, dx^j + V^{-2}(dx^5)^2 \tag{4}$$

Two consequences follow immediately from (3). First, since $\bar{g}^{ij}\bar{R}_{ij} = \bar{R} - \bar{R}_5{}^5$, transvection of (3) with $\bar{g}^{ij}$ shows that

$$\bar{R}_{55} = 0 \tag{5}$$

Second, we contemplate the Bianchi identity

$$\bar{R}^{\nu}_{\mu|\nu} - \tfrac{1}{2}\bar{R}_{|\mu} = 0 \tag{6}$$

subscripts following a bar indicating covariant derivatives in $\bar{V}_5$. Written out in full, (6) reads

$$\bar{R}^j_{i,j} - \bar{\Gamma}^k_{ij}\bar{R}_k{}^j - \bar{\Gamma}^5_{5i}\bar{R}_5{}^5 + \bar{\Gamma}^k_{kj}\bar{R}_i{}^j + \bar{\Gamma}^5_{5j}\bar{R}_i{}^j - \tfrac{1}{2}\bar{R}_{,i} = 0$$

Using (3) and (5) this reduces to

$$\bar{R}_{,i} = \bar{\Gamma}^5_{5i}\bar{R} = -V^{-1}V_{,i}\bar{R}$$

It follows that, to within an irrelevant constant factor,

$$\bar{R} = V^{-1} \tag{7}$$

Let unbarred quantities be taken as defined on the V_4 whose metric tensor is g_{ij}. Then the components of the Ricci tensor $\bar{R}_{\mu\nu}$ may be expressed in terms of those of R_{ij}, of g_{ij}, and of V and its concomitants. One finds that

$$\bar{R}_{ij} = R_{ij} + V^{-1}V_{;ij} + g_{ij}V^{-1}\Box V \tag{8a}$$

$$R_{i5} = 0 \tag{8b}$$

$$\bar{R}_{55} = -V^{-5}\Box V \tag{8c}$$

$$\bar{R} = V^{-2}R + 4V^{-3}\Box V \tag{8d}$$

Equations (5) and (8c) now imply that

$$\Box V = 0 \tag{9}$$

In turn, (7), (8d), and (9) imply the somewhat surprising relation

$$R\bar{R} = 1 \tag{10}$$

Therefore, recalling (7),

$$V = R \tag{11}$$

and (3) and (8a) together then show that

$$R_{ij} + R^{-1}R_{;ij} - \tfrac{1}{2}g_{ij}R = 0 \tag{12}$$

which is just equation (1a).